

On main eigenvalues of certain graphs

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Abstract

An eigenvalue of the adjacency matrix of a graph is said to be *main* if the all-1 vector is not orthogonal to the associated eigenspace. In this work, we approach the main eigenvalues of some graphs. The graphs with exactly two main eigenvalues are considered and a relation between those main eigenvalues is presented. The particular case of harmonic graphs is analyzed and they are characterized in terms of their main eigenvalues without any restriction on its combinatorial structure. We give a necessary and sufficient condition for a graph G to have $-1 - \lambda_{\min}$ as an eigenvalue of its complement, where λ_{\min} denotes the least eigenvalue of G . Also, we prove that among connected bipartite graphs, $K_{r,r}$ is the unique graph for which the index of the complement is equal to $-1 - \lambda_{\min}$. Finally, we characterize all paths and all double stars (trees with diameter three) for which the smallest eigenvalue is non-main. Main eigenvalues of paths and double stars are identified.

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1 Introduction and preliminaries

In 1970, Cvetković [3] introduced the concept of main eigenvalue of a graph, that is, the ones for which the associated eigenspaces are non-orthogonal to the vector whose entries equal 1. One year later, the same author [8] related the main eigenvalues directly to the number of walks in a graph. It is well known that a graph is regular if and only if it has only one main eigenvalue [20] but the characterization of graphs with exactly $s > 1$ main eigenvalues is a problem proposed in [4] which remains open. The graphs with just two main eigenvalues, have been studied in several papers [18],[12], [22], [14], [19], [20]. Namely, in [20], a complete survey on main eigenvalues of a graph (up to 2007) is given and the graphs with exactly two main eigenvalues deserve particular attention. New results on these graphs appear in more recent publications [2], [21], [16].

In Section 2, some properties of the graphs with exactly two main eigenvalues are studied. Namely, a relation between the two main eigenvalues is deduced and the harmonic graphs are characterized by its main spectral properties, without any restriction on its combinatorial structure.

Em 1971, Cvetković [8] started to investigate the relation between the main eigenvalues of a graph and the eigenvalues of its complement. This subject was also approached in [12], [22] and [1]. In [1], such relations were investigated seeking estimates for the maximal size of regular induced subgraphs in the context of convex quadratic programming applied to graphs. At the end of the paper, three questions are raised. The first wonders about the existence of a connected graph G whose complement \overline{G} has a main eigenvalue between its first eigenvalue (*index*) and $-1 - \lambda_{\min}(G)$, where $\lambda_{\min}(G)$ denotes the least eigenvalue of G . The second raises the possibility of characterizing graphs G whose spectrum of \overline{G} contains $-1 - \lambda_{\min}(G)$ as a main eigenvalue. The third question approaches the characterization of connected graphs for which the least eigenvalue is non-main. To answer these questions was the first motivation for the results of Sections 3 and 4.

In Section 3 we answer, in the negative form, to the first question posed in [1]. The largest and the second largest eigenvalues of the complement of a graph are also analyzed and we conclude that $-1 - \lambda_{\min}(G)$ belongs to its spectrum if and only if it coincides with its second largest eigenvalue. We show that, among all connected bipartite graphs, the balanced complete bipartite graphs $K_{r,r}$ are those whose respective complements contain $-1 -$

$\lambda_{\min}(G)$ as a main eigenvalue.

In Section 4 we determine the *main spectrum* (the set of distinct main eigenvalues) of a path with n vertices and conclude that the least eigenvalue of such graph is non-main if and only if n is even. Finally, we conclude that among the trees of diameter three (double stars) only the balanced ones have the least eigenvalue non-main. On the other hand, the main eigenvalues of an arbitrary double star are determined and it is shown that their main spectra has cardinality four when they are not balanced.

Throughout this paper, unless otherwise stated, G denotes a simple graph of order n with edge set $E(G)$ and vertex set $V(G) = \{1, \dots, n\}$. The edges with end-vertices i and j are simply denoted ij and the complement of the graph G is denoted \overline{G} . The *adjacency matrix* of G , $\mathbf{A} = [a_{ij}]$, is the $n \times n$ matrix for which the entries are $a_{ij} = 1$ if $ij \in E(G)$, and 0 otherwise. The eigenvalues of \mathbf{A} are also called the *eigenvalues of G* . We write $\text{Spec}(G)$ for the multi-set of eigenvalues of G . The characteristic polynomial of \mathbf{A} is called the *characteristic polynomial of G* . Unless otherwise stated, the eigenvalues of G are considered in non-increasing order, that is, $\lambda_{\max} = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n = \lambda_{\min}$. When necessary, we write $\mathbf{A}(G)$ instead of \mathbf{A} and $\lambda_i(G)$ instead of λ_i , for $i \in \{1, 2, \dots, n\}$. The eigenvectors associated to the eigenvalues of G are also called the *eigenvectors of G* and the eigenspace associated to the eigenvalue λ of G is denoted $\varepsilon_G(\lambda)$. An eigenvector associated to the largest eigenvalue of G is usually called *principal eigenvector* of G . The all one $n \times n$ matrix is denoted \mathbf{J} and \mathbf{j} denotes a column of the matrix \mathbf{J} . An eigenvalue λ of G is said to be a *main eigenvalue* if there is an associated eigenvector \mathbf{v} which is not orthogonal to \mathbf{j} . Otherwise, we say that λ is a *non-main* eigenvalue of G . Notice that for every graph G , its largest eigenvalue λ_1 is a main eigenvalue. In particular, when G is r -regular with spectrum $\lambda_1 = r, \lambda_2, \dots, \lambda_n$, all eigenvalues but λ_1 are non-main [5].

For the basic notions and notation from spectral graph theory not herein defined the reader is referred to [7]. For further mention, we also recall the following consequence of the theorem of Perron-Frobenius (see [6]).

Theorem 1 *A graph G is connected if and only if its largest eigenvalue is simple and there exists an associated eigenvector for which all the coordinates are positive.*

2 Graphs with exactly two main eigenvalues

In this section, a relation between the main eigenvalues of a graph with exactly two main eigenvalues is presented and special attention is given to the harmonic non regular graphs which are characterized using just their main spectral properties. Furthermore, the bipartite graphs with just two main eigenvalues are also analyzed. From now on, $\mathbf{d}_G = [d_1, \dots, d_n]^\top$ denotes the degrees vector of G , where d_i is the degree of the vertex $i \in V(G)$.

2.1 A relation between the two main eigenvalues

Proposition 2 *If G is a graph with m edges and exactly two main (distinct) eigenvalues λ_1 and λ_i then*

$$\lambda_i = \frac{\sum_{i \in V(G)} d_i^2 - 2m\lambda_1}{2m - n\lambda_1}. \quad (1)$$

Proof. It is immediate that there are scalars α and β and orthonormal eigenvectors \mathbf{u} and \mathbf{v} of \mathbf{A} associated to λ_1 and λ_i , respectively, such that $\mathbf{j} = \alpha\mathbf{u} + \beta\mathbf{v}$ and then $\mathbf{A}\mathbf{j} = \mathbf{d}_G = \alpha\lambda_1\mathbf{u} + \beta\lambda_i\mathbf{v}$. Then, it follows

$$\begin{aligned} \mathbf{j}^\top \mathbf{j} &= \alpha^2 + \beta^2 = n & (i) \\ \mathbf{j}^\top \mathbf{d}_G &= \lambda_1 \alpha^2 + \lambda_i \beta^2 = 2m & (ii) \\ \mathbf{d}_G^\top \mathbf{d}_G &= \lambda_1^2 \alpha^2 + \lambda_i^2 \beta^2 = \sum_{i \in V(G)} d_i^2 & (iii). \end{aligned}$$

Let us consider two cases (a) $\lambda_i = -\lambda_1$ and (b) $\lambda_i > -\lambda_1$.

(a) Replacing λ_i by $-\lambda_1$ in (iii), it follows that $\lambda_1^2 (\alpha^2 + \beta^2) = \sum_{i \in V(G)} d_i^2$.

Therefore, applying (i), we obtain $\lambda_1^2 = \frac{\sum_{i \in V(G)} d_i^2}{n}$ which in this case is equivalent to (1).

(b) Considering the pairs of equations (i)-(ii) and (i)-(iii), we obtain

$$(i)-(ii) \begin{cases} \alpha^2 = \frac{2m - n\lambda_i}{\lambda_1 - \lambda_i} \\ \beta^2 = \frac{n\lambda_1 - 2m}{\lambda_1 - \lambda_i} \end{cases} \quad (i)-(iii) \begin{cases} \alpha^2 = \frac{\sum_{j \in V(G)} d_j^2 - n\lambda_i^2}{\lambda_1^2 - \lambda_i^2} \\ \beta^2 = \frac{n\lambda_1^2 - \sum_{j \in V(G)} d_j^2}{\lambda_1^2 - \lambda_i^2} \end{cases}.$$

Therefore, from $2m - n\lambda_i = \frac{\sum_{i \in V(G)} d_i^2 - n\lambda_i^2}{\lambda_1 + \lambda_i}$, the equality (1) follows.

□

Let us call the graphs with the same main eigenvalues *co-main-spectral* graphs. Despite the relation (1) between the index of G and the other main eigenvalue, in [2] infinite families of non isomorphic co-main-spectral graphs with exactly two main eigenvalues were presented. For instance, the bidegreed graphs (that is, graphs where all vertices have one of two possible degrees) H_k^q obtained from a connected k -regular graph H_k of order p , after attaching $q \geq 1$ pendent vertices to each vertex of H_k (then the order of H_k^q is $n = (q + 1)p$) were considered. All of these graphs (independently of p) have exactly the two main eigenvalues $\lambda_{1,i}(H_k^q) = \frac{k \pm \sqrt{k^2 + 4q}}{2}$. If $k = 2$, that is, H_2 is the cycle C_p , then $\forall p \geq 3$

$$\lambda_1(H_2^q) = 1 + \sqrt{1 + q} \quad \text{and} \quad \lambda_i(H_2^q) = 1 - \sqrt{1 + q},$$

are its two main eigenvalues. Notice that for $p = 3, 4, \dots$ we obtain an infinite sequence of co-main-spectral graphs of increasing order equal to $(q + 1)p$ (see [2, Fig. 2]). In spite of this, taking into account that H_2^q has $m = n$ edges and $\sum_{i \in V(H_2^q)} d_i^2 = p(q + 2)^2 + pq$ it is easy to check that the equality (1) holds.

The next immediate corollary of Proposition 2 it will be useful for the study of harmonic graphs.

Corollary 3 *If $\lambda_i = 0$, then $\lambda_1 = \frac{\sum_{i \in V(G)} d_i^2}{2m}$.*

2.2 Harmonic graphs

A graph H is said to be *harmonic* when \mathbf{d}_H is an eigenvector associated to a (necessarily) integer eigenvalue, that is, if there is a positive integer ℓ such that $\mathbf{A}\mathbf{d}_H = \ell\mathbf{d}_H$. It is immediate that every regular graph is harmonic. The harmonic graphs were introduced in [11], [10]. In [19], such a graph without isolated vertices is called *pseudo-regular graph* and it is defined as being a graph H such that $\sum_{j \in N_H(i)} \frac{d_H(j)}{d_H(i)}$ is constant for every $i \in V(H)$.

The particular case of harmonic trees was studied in [11], where the author consider the trees \mathcal{T}_ℓ , with $\ell \geq 2$, such that one of its vertices v has degree $\ell^2 - \ell + 1$, while every neighbor of v has degree ℓ and all the remaining vertices have degree 1. He proved that these are the unique harmonic trees. These trees are among the trees with two main eigenvalues which have been characterized in [15] (see also [14]).

Theorem 4 ([15]) *The stars, the balanced double stars and the harmonic trees \mathcal{T}_ℓ , for $\ell \geq 2$, are the unique trees with exactly two main eigenvalues.*

Lemma 5 *If H is a bipartite harmonic graph with at least one edge and largest eigenvalue λ_1 , then $-\lambda_1$ is a non-main eigenvalue of H .*

Proof. Let us consider that the bipartite harmonic graph H has $q \geq 1$ connected nontrivial components H_1, \dots, H_q and H_{q+1}, \dots, H_{q+p} trivial components, with $p \geq 0$. It is immediate that each H_k of order n_k is a connected bipartite harmonic subgraph with the same largest eigenvalue λ_1 and the same simple least eigenvalue $-\lambda_1$. Assuming that $V(H_k)$ admits the bipartition S_k and T_k such that each edge of H_k has one end-vertex in S_k and the other in T_k , then the vectors $\begin{pmatrix} d_{S_k} \\ d_{T_k} \end{pmatrix}$ and $\begin{pmatrix} -d_{S_k} \\ d_{T_k} \end{pmatrix}$, where d_{S_k} and d_{T_k} denote the subvectors of degrees of the vertices in S_k and T_k , are the principal eigenvector and the eigenvector associated to $-\lambda_1$, respectively, of H_k , for $k = 1, \dots, q$. The vectors

$$\hat{u}_k^T = (0, \dots, 0, d_{S_k}^T, d_{T_k}^T, 0, \dots, 0, 0, \dots, 0) \quad (2)$$

$$\hat{v}_k^T = (0, \dots, 0, -d_{S_k}^T, d_{T_k}^T, 0, \dots, 0, 0, \dots, 0) \quad (3)$$

where the last p zero coordinates correspond to the p trivial components, the $k-1$ zeros on the left of $d_{S_k}^T$ and the $q-k$ zeros on the right of $d_{T_k}^T$ correspond to the vertices in the components $H_1, \dots, H_{k-1}, H_{k+1}, \dots, H_q$, respectively, are also a principal eigenvector and the eigenvector associated to $-\lambda_1$, respectively, of H . Therefore, since for each component H_k the sum of the degrees of the vertices in S_k is equal to the sum of the degrees of the vertices in T_k , it follows that the vectors in (3) are all orthogonal to the all one vector. Since every vector of the eigenspace associated to $-\lambda_1$ is a linear combination of those vectors in (3), $-\lambda_1$ is non-main. \square

Nikiforov in [19, Th. 8] proved that every main eigenvalue of an harmonic graph H belongs to the set $\{-\lambda_1, 0, \lambda_1\}$. It is also stated in [19, Th. 8] that if H is a graph without a bipartite component such that all main eigenvalues are in $\{-\lambda_1, 0, \lambda_1\}$, then it is harmonic. A similar result for connected graphs is obtained in [20, Pr. 3.3], using a different approach. The next proposition gives a spectral characterization of harmonic graphs without any restriction regarding their combinatorial structure.

Proposition 6 *A graph H is harmonic if and only if every main eigenvalue of H belongs to the set $\{0, \lambda_1\}$.*

Proof. If H is harmonic, as direct consequence of Lemma 5 and Theorem 8 in [19], it follows that every main of its eigenvalues are in $\{0, \lambda_1\}$. Conversely, let us consider that the main eigenvalues of H are in $\{0, \lambda_1\}$. If H has

only one main eigenvalue, then H is regular and the result follows. Otherwise, assuming that $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a basis for $\varepsilon_H(0)$ and $\{\mathbf{u}_1, \dots, \mathbf{u}_q\}$ is a basis for $\varepsilon_H(\lambda_1)$, it follows that $\mathbf{j} = \sum_{i=1}^p \alpha_i \mathbf{u}_i + \sum_{j=1}^q \beta_j \mathbf{v}_j$ for some scalars $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_q$ and $\mathbf{d}_H = A_H \mathbf{j} = \lambda_1 \sum_{i=1}^p \alpha_i \mathbf{u}_i$. Therefore, $\mathbf{d}_H \in \varepsilon_H(\lambda_1)$. \square

The following proposition gives an alternative characterization of harmonic graphs.

Proposition 7 *A graph G with m edges is harmonic if and only if $\lambda_1 = \frac{\sum_{i \in V(G)} d_i^2}{2m}$ and it has no more than two main eigenvalues.*

Proof. Suppose that G is harmonic. By Proposition 6, all its main eigenvalues are in $\{0, \lambda_1\}$ and we have two cases: (i) G is regular, with degree say k , and then $\lambda_1 = k$ is the unique main eigenvalue or (ii) G is non regular and then it has two main eigenvalues.

$$(i) \quad \lambda_1 = k = \frac{nk^2}{nk} = \frac{\sum_{i \in V(G)} d_i^2}{2m}.$$

$$(ii) \quad \text{By Corollary 3, it follows that } \lambda_1 = \frac{\sum_{i \in V(G)} d_i^2}{2m}.$$

Conversely, assume that $\lambda_1 = \frac{\sum_{i \in V(G)} d_i^2}{2m}$ and G has no more than two main eigenvalues. If G is regular then the conclusion is immediate. Else, by Proposition 2, the main eigenvalues of G , $\lambda_i(G)$ and λ_1 , are related by the equality (1). Replacing λ_1 in (1) by $\frac{\sum_{i \in V(G)} d_i^2}{2m}$ it follows that the main eigenvalues of G are in $\{0, \lambda_1\}$. Therefore, by Proposition 6, the result follows. \square

3 The largest and the second largest eigenvalues of the complement of a graph

From now on, we consider the all distinct eigenvalues μ_1, \dots, μ_s , $1 \leq s \leq n$, of the graph G having the respective associated eigenspace not orthogonal to the vector \mathbf{j} as the *main* eigenvalues of G and the remaining distinct eigenvalues μ_{s+1}, \dots, μ_p , $s+1 \leq p \leq n$, as the *non-main* eigenvalues. The set of distinct main eigenvalues of G is herein called the *main spectrum* of G and it is denoted $MainSpec(G)$. Therefore, $Spec(G) = \{\mu_1^{[q_1]}, \dots, \mu_s^{[q_s]}, \mu_{s+1}^{[q_{s+1}]}, \dots, \mu_p^{[q_p]}\}$, where $\mu_j^{[q_j]}$ means that the eigenvalue μ_j has multiplicity q_j .

Before to proceed, it is worth to recall the following theorem.

Theorem 8 ([8]) *MainSpec(G) and MainSpec(\overline{G}) have the same number of elements. Furthermore, if $\lambda \in \text{MainSpec}(G)$ and $\overline{\lambda} \in \text{MainSpec}(\overline{G})$, then $\lambda + \overline{\lambda} \neq -1$.*

Taking into account this theorem and the definition of main/non-main eigenvalue it is immediate to obtain the basic results stated in the next proposition partially proved in [12].

Proposition 9 *Consider a graph G and $\lambda \in \text{Spec}(G)$. Then the following assertions are equivalent:*

1. *the eigenvalue λ is non-main or it is main with multiplicity greater than 1;*
2. *there is some eigenvector \mathbf{v} of G associated to λ such that $\mathbf{j}^\top \mathbf{v} = 0$;*
3. *the scalar $-1 - \lambda$ belongs to $\text{Spec}(\overline{G})$.*

As direct consequence of this proposition, we may note that a necessary and sufficient condition for a simple eigenvalue λ of a graph G to be non-main is $-1 - \lambda$ to be an eigenvalue of \overline{G} (see [12]).

Furthermore, we also may conclude the following corollary of Proposition 9.

Corollary 10 *If $-1 - \lambda(G)$ is a simple eigenvalue of \overline{G} then it is non-main.*

Proof. It follows from Proposition 9, in view of the known relation $\mathbf{A}(\overline{G}) = \mathbf{J} - \mathbf{I}_n - \mathbf{A}(G)$. \square

Now, it is worth to recall the following consequence of Weyl's inequalities which proof can be found in [6]:

$$\lambda_2(\overline{G}) \leq -1 - \lambda_n(G) \leq \lambda_1(\overline{G}). \quad (4)$$

The relations (4) furnish a (negative) answer to the question raised in [1] about the existence of a graph G for which the complement \overline{G} has an eigenvalue less than its index and greater than $-1 - \lambda_n(G)$.

Proposition 11 *If G is a graph of order n , then \overline{G} has no eigenvalue belonging to the open interval $(-1 - \lambda_n(G), \lambda_1(\overline{G}))$.*

The inequalities (4) motivate us to consider graphs G for which $-1 - \lambda_n(G)$ is an eigenvalue of \overline{G} . We have two cases: (a) $\lambda_1(\overline{G}) = -1 - \lambda_n(G)$ and (b) $\lambda_2(\overline{G}) = -1 - \lambda_n(G)$.

In the case (a), we have that $-1 - \lambda_n(G) = \lambda_1(\overline{G})$ is a main eigenvalue of \overline{G} . Therefore, Theorem 8 guarantees that $\lambda_n(G)$ is a non-main eigenvalue, since $-1 = \lambda_n(G) + (-1 - \lambda_n(G))$. In fact, regarding the equality (a), we may establish the following proposition.

Proposition 12 *Let G be a graph of order n . Then $\lambda_1(\overline{G}) = -1 - \lambda_n(G)$ if and only if $\lambda_n(G)$ is non-main and the multiplicity of $\lambda_1(\overline{G})$ is greater than one.*

Proof. If $\lambda_1(\overline{G}) = -1 - \lambda_n(G)$, $\lambda_n(G)$ is non-main and from Proposition 9, $-1 - \lambda_n$ has an eigenvector \mathbf{v}_1 such that $\mathbf{v}_1 \in \varepsilon_G(\lambda_n)$ and $\mathbf{j}^\top \mathbf{v}_1 = 0$. On the other hand (by Perron-Frobenius theorem), there is an eigenvector \mathbf{v}_2 , associated to $\lambda_1(\overline{G})$, with nonnegative entries and then $\mathbf{j}^\top \mathbf{v}_2 \neq 0$. Therefore, \mathbf{v}_1 and \mathbf{v}_2 are linearly independent. This implies that the multiplicity of $\lambda_1(\overline{G})$ is greater than 1. Conversely, if $\lambda_n(G)$ is non-main then $-1 - \lambda_n(G) \in \text{Spec}(\overline{G})$ by Proposition 9. Since the multiplicity of $\lambda_1(\overline{G})$ is greater than one, from (4) the result follows. \square

According to Theorem 1 and Proposition 12, when $\lambda_1(\overline{G}) = -1 - \lambda_n(G)$ it follows that $\lambda_n(G)$ is non-main and \overline{G} is disconnected. On the other hand, for the case (b) we have:

Proposition 13 *Let G be a graph of order n . Then $\lambda_2(\overline{G}) = -1 - \lambda_n(G) < \lambda_1(\overline{G})$ if and only if $\lambda_n(G)$ is main with multiplicity greater than one or it is non-main and $\lambda_1(\overline{G})$ is simple.*

The inequalities in (4) and Propositions 12 and 13 allow us to conclude that $-1 - \lambda_n(G) \in \text{Spec}(\overline{G})$ if and only if $\lambda_2(\overline{G}) = -1 - \lambda_n(G)$.

From Corollary 10, for an arbitrary graph G of order n such that $\lambda_n(G)$ is a simple eigenvalue, we have that $\lambda_n(G)$ is non-main if and only if $-1 - \lambda_n(G)$ is an eigenvalue of \overline{G} . Since the least eigenvalue of a connected bipartite graph is simple, for these graphs we may conclude the following:

- (a) For a connected bipartite graph G , $\lambda_1(\overline{G}) = -1 - \lambda_n(G)$ if and only if $\lambda_n(G)$ is non-main and $\lambda_1(\overline{G})$ has multiplicity greater than one.
- (b) If G is connected and bipartite then $\lambda_2(\overline{G}) = -1 - \lambda_n(G) < \lambda_1(\overline{G})$ if and only if $\lambda_n(G)$ is non-main and $\lambda_1(\overline{G})$ is simple.

The next result gives a combinatorial characterization of bipartite graphs G of order n for which $\lambda_1(\overline{G}) = -1 - \lambda_n(G)$.

Theorem 14 *Let G be a bipartite graph of order n . Then $\lambda_1(\overline{G}) = -1 - \lambda_n(G)$ if and only if G is complete (bipartite) and balanced.*

Proof. Let us consider a bipartite graph G with vertex set $V = V_1 \dot{\cup} V_2$, where $|V_1| = r$ and $|V_2| = s$. If $\lambda_1(\overline{G}) = -1 - \lambda_n(G)$ then (a) above implies \overline{G} is disconnected, and thus $\overline{G} = K_r \dot{\cup} K_s$. Since $\lambda_1(G)$ is a multiple eigenvalue then $r = s$. Conversely, if $G = K_{s,s}$, for some positive integer s , then $\overline{G} = \overline{K_{s,s}}$ is a disconnected graph with two components which are complete graphs with s vertices. It follows that $\lambda_n(G) = -s$ and $\lambda_1(\overline{G}) = s - 1$ and therefore, $\lambda_1(\overline{G}) = -\lambda_n(G) - 1$. \square

4 Main spectra of paths and double stars.

Concerning the third question of [1], we may note that among the connected graphs for which the least eigenvalue is non-main we can count the harmonic graphs (see the Proposition 6) which includes the regular graphs. In this section, the graphs with non-main least eigenvalue of two families of trees are characterized. We start by determining the paths with non-main least eigenvalue. For sake of completeness, we determine the main spectrum of an arbitrary path.

It is worth to recall the following lemma which can be found in [5] (the eigenvectors are described in [17]).

Lemma 15 ([5],[17]) *Let \mathcal{P}_n be the path on n vertices. Then its eigenvalues are simple and given by $\lambda_j(\mathcal{P}_n) = 2 \cos\left(\frac{j\pi}{n+1}\right)$, $1 \leq j \leq n$. Each of these eigenvalues λ_j has an associated eigenvector with entries $\mathbf{v}_i^{(j)} = \sin\left(i \frac{j\pi}{n+1}\right)$, for $i \in \{1, 2, \dots, n\}$.*

Theorem 16 *For $n \geq 2$ and $1 \leq j \leq n$, λ_j is a non-main eigenvalue of the path \mathcal{P}_n if and only if j is even. In particular, the least eigenvalue of \mathcal{P}_n is non-main if and only if n is even.*

Proof. Let us fix j , $1 \leq j \leq n$. For the λ_j -eigenvector $\mathbf{v}^{(j)} = (\mathbf{v}_1^{(j)}, \dots, \mathbf{v}_n^{(j)})^\top$ we have $\lambda \mathbf{v}_i^{(j)} = \sum_{t \sim i} \mathbf{v}_t^{(j)}$, whence $\lambda \sum_i \mathbf{v}_i^{(j)} = \sum_i d_i \mathbf{v}_i^{(j)} = 2 \sum_i \mathbf{v}_i^{(j)} -$

$\mathbf{v}_1^{(j)} - \mathbf{v}_n^{(j)}$. From Lemma 15, $\lambda_j \neq 2$ and then $\sum_i \mathbf{v}_i^{(j)} = 0$ if and only if $\mathbf{v}_1^{(j)} + \mathbf{v}_n^{(j)} = 0$. Since $\mathbf{v}_1^{(j)} + \mathbf{v}_n^{(j)} = 2 \sin\left(\frac{j\pi}{2}\right) \cos\left(\frac{(n-1)j\pi}{2(n+1)}\right)$, we may verify that λ_j is a non-main eigenvalue if and only if j is even. In fact, $\cos\left(\frac{(n-1)j\pi}{2(n+1)}\right) = 0$ if and only if $\frac{(n-1)j\pi}{2(n+1)} = \frac{\pi}{2} + k\pi$, for $k \in \mathbb{N}$. From this, we have that $1 + 2k < j \leq n$. Also, it holds that $n = \frac{j+(1+2k)}{j-(1+2k)}$, which implies $n < \frac{2n}{j-(1+2k)}$ and then, $j < 2k + 3$. Thus $1 + 2k < j < 2k + 3$, that is, j is even. The another case is straightforward. \square

Corollary 17 *The path \mathcal{P}_n on n vertices has $\lceil \frac{n}{2} \rceil$ main eigenvalues, where $\lceil x \rceil$ denotes the least integer no less than x .*

The following result characterizes the semi-regular bipartite graphs in terms of theirs main eigenvalues.

Theorem 18 ([20]) *A non-trivial connected graph G is semi-regular bipartite if and only if its main eigenvalues are only $\lambda_1(G)$ and $-\lambda_1(G)$.*

Combining Theorem 18 with Proposition 2, it follows that when G is a connected semi-regular bipartite graph of order n , $\lambda_1(G) = \sqrt{\frac{\sum_{i \in V(G)} d_i^2}{n}}$. This is a known result obtained in [13] where it was stated that if a graph H has order n , then $\sum_{i \in V(H)} d_i^2 \leq \lambda_1^2(H)n$ and the equality holds if and only if H is a semi-regular bipartite graph.

Since diameter-2 trees (the stars) are connected semi-regular bipartite graphs, there exists no diameter-2 tree with non-main least eigenvalue. Regarding diameter-3 trees, it should be noted that these trees (the double stars) are not semi-regular bipartite graphs and then, combining Theorems 18 and 4, we may conclude that the least eigenvalue of a balanced double star is non-main. On the other hand, we claim that there are no non-balanced double stars with least eigenvalue non-main.

In order to prove our assertion, we first remember that a *walk* in G is a sequence i_0, i_1, \dots, i_r of vertices in G such that i_t is adjacent to i_{t+1} , for $0 \leq t \leq r - 1$. The *length* of a walk is its number of edges. For a square matrix \mathbf{B} , the *walk-matrix* of \mathbf{B} is given by $\mathbf{W}(\mathbf{B}) = [\mathbf{j} \ \mathbf{B} \mathbf{j} \ \mathbf{B}^2 \mathbf{j} \ \dots \ \mathbf{B}^{n-1} \mathbf{j}]$. In the particular case $\mathbf{B} = \mathbf{A}$, the adjacency matrix of G , $\mathbf{W}(\mathbf{A}) = [w_{ij}]$ is such that w_{ij} gives the number of walks in G of length j starting at vertex i , $1 \leq i \leq n$ and $1 \leq j \leq n - 1$, and then it is called the *walk-matrix* of G . We also recall that a partition π of the vertex set $V(G)$ of the graph G is *equitable* when, given two cells V_i and V_j of π , there is a constant m_{ij} such

each vertex $v \in V_j$ has exactly m_{ij} neighbors in V_j . The matrix $\mathbf{M} = [m_{ij}]$ is called the *divisor of G* with respect to π . It is known (Theorem 3 of [4]), that the main eigenvalues of G are eigenvalues of \mathbf{M} . A fundamental result on the number of eigenvalues of a graph is the following

Theorem 19 ([12]) *The rank of the walk-matrix of G is equal to the number of its main eigenvalues.*

It was recently proved ([23], Lemma 2.4) that the number of main eigenvalues of G is equal to the rank of the walk-matrix $\mathbf{W}(\mathbf{M})$ of \mathbf{M} .

Theorem 20 *Let T be a double star with n vertices. Then its least eigenvalue is non-main if and only if T is balanced.*

Proof. Let $T = T(k, s)$ be a double star of order $n = k + s + 2$ whose vertices are labeled as in Figure 1. Let us consider $V_1 = \{1\}$, $V_2 = \{2\}$,

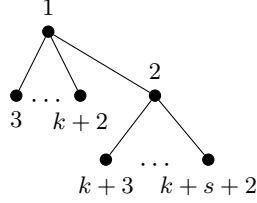


Figure 1: Double star $T(k, s)$.

$V_3 = \{3, \dots, k+2\}$ and $V_4 = \{k+3, \dots, k+s+2\}$. Then $V_1 \dot{\cup} V_2 \dot{\cup} V_3 \dot{\cup} V_4$ is an equitable partition of $V(T)$ with associated divisor

$$\mathbf{M} = [m_{ij}] = \begin{bmatrix} 0 & 1 & k & 0 \\ 1 & 0 & 0 & s \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

for which the walk-matrix is

$$\mathbf{W}(\mathbf{M}) = \begin{bmatrix} 1 & k+1 & k+s+1 & s+k^2+2k+1 \\ 1 & s+1 & k+s+1 & s^2+2s+k+1 \\ 1 & 1 & k+1 & k+s+1 \\ 1 & 1 & s+1 & k+s+1 \end{bmatrix}.$$

It can be verified that $\det \mathbf{W}(\mathbf{M}) = -ks(s-k)^2$, which is equal to zero if and only if $s = k$. Since \mathbf{M} has characteristic polynomial $q(x) = x^4 - (k +$

$s + 1)x^2 + ks$ and, according to [9], the characteristic polynomial of T is $p(x) = x^{s-1}x^{k-1}(x^4 - x^2(k + s + 1) + ks)$, we conclude that in case $k \neq s$ the four non-zero eigenvalues of the graph $T = T(k, s)$ are main. In particular, it follows that λ_n is a main eigenvalue (clearly, the others are λ_1 , λ_2 and λ_{n-1}). Considering that the case $k = r$ is already known, the assertion is proved. \square

By combining Theorem 8, Proposition 13 and Theorems 16 and 20 we may conclude immediately the next corollary.

Corollary 21 *If the graph G is a path (respectively, a balanced double star) on n vertices then its complement \overline{G} has $\lceil \frac{n}{2} \rceil$ (resp., two) main eigenvalues and the second largest eigenvalue of \overline{G} is equal to $-1 - \lambda_n(G)$.*

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